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An Asymptotic Relation for the Zeros of Bessel Functions*

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We define the function $j_{\nu\kappa}$ for all real $\kappa > 0$ as follows: for $\kappa = 1, 2,...$ the $j_{\nu\kappa}$ denotes the kth positive zero of the Bessel function $J_\nu(z)$ of first kind and for $k-1 < \kappa < k$, $j_{\nu\kappa}$ denotes the kth positive zero of the cylinder Bessel function $C_\nu(z) = \cos\alpha\, J_\nu(z) - \sin\alpha\, Y_\nu(z)$ with $\alpha = (k-\kappa)\pi$ (see [2]), where $Y_\nu(x)$ is the Bessel function of second kind. We introduce the function $\iota(x)$ for $x > -1$,

$$\iota(x) = \lim_{\kappa \to \infty} \frac{j_{\kappa x, \kappa}}{\kappa}$$

and we prove, among other things, the inequality $j_{\nu\kappa} < \kappa\iota(\nu/\kappa)$. Moreover, we find the first three terms of the asymptotic expansion of $\iota(x)$, for large values of x and other properties of this function.

1. Introduction

For $\nu > 0$, we call $j_{\nu k}$ and $c_{\nu k}$ the kth positive zeros of the Bessel function of the first kind $J_\nu(x)$ and of the general cylinder function, respectively; the latter being defined by

$$C_\nu(x) = \cos\alpha\, J_\nu(x) - \sin\alpha\, Y_\nu(x)$$

where α is fixed, $0 \leqslant \alpha < \pi$ and $Y_\nu(x)$ is the Bessel function of second kind.

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The definitions can be extended to negative values of ν in such a way that $c_{\nu k}$ varies continuously with ν, $c_{\nu k} \to 0$ where $\nu \to \alpha/\pi - k$ and on the interval

$$\frac{\alpha}{\pi} - k < \nu < \frac{\alpha}{\pi} - k + 1$$

$c_{\nu k}$ is the first positive zero of $C_\nu(x)$ [7, p. 508].

It is useful to define the function $j_{\nu\kappa}$ for all $\kappa > 0$ as in [2].

The function $c_{\nu k}$ satisfies the differential equation [7, p. 508]

$$\frac{d}{d\nu} c_{\nu k} = 2c_{\nu k} \int_0^\infty K_0(2c_{\nu k} sht)\, e^{-2\nu t}\, dt \tag{1.1}$$

where $K_0(x)$ is the modified Bessel function of order zero. This property suggests the following generalization: Let $j_{\nu\kappa}$ be the solution of the differential equation

$$\frac{d}{d\nu} j = 2j \int_0^\infty K_0(2jsht)\, e^{-2\nu t}\, dt \tag{1.2}$$

for $\kappa > 0$ with the boundary condition

$$\lim_{\nu \to -\kappa + 0} j(\nu) = 0. \tag{1.2$'$}$$

Then for $\kappa = 1, 2, \ldots$ we obtain the known functions $j_{\nu k}$. If $k - 1 < \kappa < k$ we have $j_{\nu\kappa} = c_{\nu k}$ with $\alpha = (k - \kappa)\pi$. It is easy to see that the right-hand side of (1.2) is Lipschitzian with respect to j for $j > 0$. Therefore the solutions for any initial value problem are unique. Such uniqueness also implies that if $0 < \kappa' < \kappa''$, then

$$j_{\nu\kappa'} < j_{\nu\kappa''}, \qquad \nu > -\kappa'.$$

The asymptotic behaviour of the zeros of Bessel functions has been investigated by many authors. For example, McMahon [7, p. 506] studied the behaviour of $c_{\nu k}$ when ν is fixed and k is large and gave the formula

$$c_{\nu k} = (k + \nu/2 - \tfrac{1}{4})\pi - \alpha + O(k^{-1}), \qquad k \to \infty. \tag{1.3}$$

The behaviour of $j_{\nu k}$ when k is fixed and ν is large has been investigated by Tricomi in [6]. He gave the result

$$j_{\nu k} = \nu + a_k \nu^{1/3} + O(\nu^{-1/3}), \qquad k = 1, 2, \ldots, \qquad \nu \to \infty, \tag{1.4}$$

where a_k is independent of ν.

The asymptotic relation (1.3) can be written, using our notations, in the form

$$j_{\nu\kappa} = (\kappa + \nu/2 - \tfrac{1}{4})\pi + O(\kappa^{-1}), \qquad \kappa \to \infty, \tag{1.5}$$

while the inequalities [7, p. 490]

$$(k + \nu/2 - \tfrac{1}{4})\pi - \alpha < c_{\nu k} < k\pi, \qquad k = 1, 2, \ldots, \qquad |\nu| < \tfrac{1}{2},$$

can be written, using the new notation,

$$(\kappa + \nu/2 - \tfrac{1}{4})\pi < j_{\nu\kappa} < \kappa\pi, \qquad |\nu| < \tfrac{1}{2}. \tag{1.6}$$

In this paper we introduce a new function $\imath(x)$ and give some properties of this function. The knowledge of these properties will give further information on the function $j_{\nu\kappa}$.

2. The Main Result

For $-1 \leqslant x < \infty$ we introduce the continuous functions $\alpha(x)$ and $\imath(x)$ as follows:

$$\frac{\sin\alpha}{\cos\alpha - (\pi/2 - \alpha)\sin\alpha} = \frac{x}{\pi} \tag{2.1}$$

$$\imath(x) = \begin{cases} \pi, & x = 0 \\ \dfrac{x}{\sin\alpha}, & x \neq 0. \end{cases} \tag{2.2}$$

We have $\alpha(0) = 0$, $\alpha'(0) = 1/\pi$ and it is not difficult to show that the function on the left-hand side of (2.1) is strictly increasing for $-\pi/2 \leqslant x \leqslant \pi/2$ and it varies between $-1/\pi$ and ∞. Thus the functions $\alpha(x)$ and $\imath(x)$ are well defined.

We observe also that $\imath(-1) = 1$, $\imath(x) > 1$ for $x \geqslant -1$.

Now we can formulate our main result.

Theorem 2.1. *Let $x > -1$ be fixed. Then the following relations hold:*

$$\lim_{\kappa \to +\infty} \frac{j_{\kappa x, \kappa}}{\kappa} = \imath(x) \tag{2.3}$$

$$j_{\nu\kappa} < \kappa\imath(\nu/\kappa), \qquad \kappa > 0, \qquad \nu > -\kappa. \tag{2.4}$$

Proof. It is useful to introduce the function $\iota_\kappa(x)$ as

$$\iota_\kappa(x) = j_{\kappa x, \kappa}/\kappa.$$

From (1.6) we have

$$\lim_{\kappa \to +\infty} \iota_\kappa(0) = \pi \tag{2.5}$$

and

$$\iota_\kappa(0) < \pi. \tag{2.6}$$

Moreover from (1.1) we get that $\iota_\kappa(x)$ satisfies the differential equation

$$\frac{d}{dx} \iota_\kappa(x) = 2\iota_\kappa \int_0^\infty K_o\left(u 2\iota_\kappa \frac{sh(u/\kappa)}{u/\kappa}\right) e^{-2xu}\, du. \tag{2.7}$$

Since $sht > t$, $t > 0$ and $K_o(x)$ is strictly decreasing, $x > 0$, we get

$$\frac{d}{dx} \iota_\kappa < 2\iota_\kappa \int_0^\infty K_o(2i_\kappa u)\, e^{-2xu}\, du = \int_0^\infty K_o(t)\, e^{-(x/\iota_\kappa)t}\, dt.$$

For the function $K_o(x)$ the following formula holds:

$$K_o(x) \sim \sqrt{\pi/2t}\, e^{-t}, \qquad t \to \infty.$$

Thus, in order to assure the convergence of the integral on the right-hand side in (2.7) the restriction $-x/\iota_\kappa(x) < 1$ is necessary.

Now consider the initial value problem for the differential equation

$$\frac{d}{dx} \hat{\iota}(x) = \int_0^\infty K_o(t)\, e^{-x/\hat{\iota}(x)}\, dt, \tag{2.8}$$

$\hat{\iota}(0) = \pi$. The right-hand side of (2.8) is Lipschitzian with respect to $\hat{\iota}$ provided $\hat{\iota} > \varepsilon > 0$, where ε is a positive number. We shall see later that we have $\hat{\iota}(x) > 1$, i.e., we can choose $\varepsilon = 1$.

Let (x_1, x_2) be the domain of existence of the solution $\hat{\iota}(x)$ of the initial value problem, with the restriction

$$\left| \frac{x}{\hat{\iota}(x)} \right| < 1. \tag{2.9}$$

Since $\hat{\iota}(0) \neq 0$, we have $x_1 < 0 < x_2$ and by using the integral formula [7, p. 388]

$$\int_0^\infty K_o(t)\, e^{-at}\, dt = \frac{\operatorname{arcos} a}{\sqrt{1 - a^2}}, \qquad |a| < 1,$$

we obtain

$$\frac{d}{dx}\hat{\imath}(x)=\frac{\arccos(x/\hat{\imath}(x))}{\sqrt{1-(x/\hat{\imath}(x))^2}}. \tag{2.10}$$

The function $\imath(x)$ defined by (2.4) is a solution of our initial value problem with

$$\frac{x}{\hat{\imath}(x)}=\sin\alpha(x) \tag{2.11a}$$

$$\hat{\imath}'(x)=\frac{\pi/2-\alpha}{\cos\alpha}. \tag{2.11b}$$

Thus by the uniqueness of the initial value problem the function $\imath(x)$ is the only solution of (2.8). Consequently the domain of existence is $(-1,+\infty)$, i.e., $x_1=-1$, $x_2=+\infty$.

To simplify the notations we write the differential equations (2.7) and (2.9) in the form

$$\imath_\kappa'=F_\kappa(\imath_\kappa, x) \tag{2.12}$$

$$\hat{\imath}=G(\hat{\imath}, x) \tag{2.13}$$

respectively, with

$$F_\kappa(y,x)<G(y,x), \qquad x\geqslant -1, \qquad y>0. \tag{2.14}$$

Moreover by the relation

$$\lim_{\kappa\to+\infty}\frac{sh(u/\kappa)}{u/\kappa}=1$$

we get

$$\lim_{\kappa\to+\infty}F_\kappa(y,x)=G(y,x). \tag{2.15}$$

To show that

$$j_{\kappa\lambda,\kappa}/\kappa=\imath_\kappa(x)<\imath(x), \qquad x\geqslant -1, \tag{2.16}$$

which is equivalent to (2.4), we introduce the function

$$f(x)=\imath(x)-\imath_\kappa(x).$$

From (2.2), (2.6) we get $\imath_\kappa(0)<\imath(0)=\pi$, i.e., $f(0)>0$, hence (2.14) implies [3, pp. 10–11] $\imath_\kappa(x)<\imath(x)$ and $f(x)>0$ for $x\geqslant 0$.

Concerning the interval $[-1, 0)$ it is $\iota(-1) = 1$ and, from (1.2), $\iota_\kappa(-1) = 0$, hence $f(-1) = 1 > 0$. Suppose that $f(x) > 0$ for some x on $(-1, 0)$, then there exists a value ξ on $(-1, 0)$ such that $f(\xi) \leqslant 0$.

Let η defined by

$$\eta = \max\{x; f(t) > 0, -1 \leqslant t < x, -1 < x < \xi\}.$$

It is clear that $f(\eta) = 0, f'(\eta) \leqslant 0$ and consequently

$$\iota'(\eta) \leqslant \iota_\kappa'(\eta), \qquad \iota(\eta) = \iota_\kappa(\eta). \tag{2.17}$$

On the other hand it follows from (2.11) that

$$\iota_\kappa'(\eta) = F_\kappa(\iota_\kappa(\eta), \eta) = F_\kappa(\iota(\eta), \eta) < G(\iota(\eta), \eta) = \iota'(\eta)$$

which contradicts (2.17). Hence the relation (2.16) is satisfied for all $x \geqslant -1$.

Finally, the formula (2.3) is a consequence of the continuous dependence of the solutions on the initial values and the relations (2.15) and (2.5). Thus the proof of the theorem is complete.

Remark 2.1. The convergence of the functions $\iota_\kappa(x) = j_{\kappa x, \kappa}/\kappa$ to $\iota(x)$ is not uniform in the neighbourhood of $x = -1$, because from (1.2) $\iota_\kappa(-1) = j_{-\kappa,\kappa} = 0$ for every $\kappa > 0$, while $\iota(-1) = 1$.

Remark 2.2. Concerning the behaviour of the function $\iota_\kappa(x)$ with respect to κ and fixed $x > -1$ we conjecture that this function increases as κ increases.

This conjecture is supported by two facts. First, by using (2.1), (2.2) we find that $\iota_\kappa(x)$ approaches $\iota(x)$ from below as $\kappa \to \infty$.

On the other hand the zeros $c_{\nu k}$ of $C_\nu(x)$ satisfy the inequalities [5, p. 167]

$$0 < \delta_1 = c_{\nu 1} \leqslant j_{\nu 1}$$
$$\delta_2 = c_{\nu 2} - c_{\nu 1} < \delta_3 = c_{\nu 3} - c_{\nu 2} < \cdots, \qquad |\nu| < \tfrac{1}{2}.$$

From these inequalities we obtain

$$c_{\nu, k+1} = \delta_1 + \delta_2 + \cdots + \delta_k + \delta_{k+1} > \delta_1 + \delta_2 + \cdots + \delta_k$$
$$+ \frac{1}{k}(\delta_1 + \delta_2 + \cdots + \delta_k) = \frac{k+1}{k} c_{\nu k}, \qquad k = 1, 2, \ldots .$$

Thus the sequence $\{c_{\nu k}/k\}_{k=1}^{\infty}$ increases and for $\alpha = 0$ and $\nu = 0$ also the sequence $j_{ok}/k = \iota_\kappa(0)$ increases as k increases.

Remark 2.3. The inequality (2.4) can be used directly to get an upper bound for $j_{\nu\kappa}$, in any case. For example, when $\nu = \kappa = 10$ we get $j_{10,10} < 10\,\iota(1) = 46.11\ldots$, while the exact value is $45.23\ldots$.

3. Properties of $\iota(x)$

In view of the concavity of $j_{\nu k}$, $k = 1, 2, \ldots$, $-k < \nu < \infty$ [1], the function $\iota_\kappa(x) = j_{kx}/k$ is concave, hence from (2.3) the function $\iota(x)$ is also concave.

We can also show this property using the derivative of $\iota'(x)$ as given by (2.11).

Now we can enunciate the main result of this section.

THEOREM 3.1. *The function $\iota(x)$ admits the following asymptotic representation*:

$$\iota(x) = x + \frac{(3\pi)^{2/3}}{2} x^{1/3} + \frac{3}{40}(3\pi)^{4/3} x^{-1/3} + O(x^{-1}), \qquad x \to \infty. \tag{3.1}$$

Moreover the inequality

$$\iota(x) > x + \frac{(3\pi)^{2/3}}{2} x^{1/3}, \qquad x > 0, \tag{3.2}$$

holds.

Proof. For the proof of (3.2) it is convenient to introduce the function

$$I(x) = \iota(x) - x - Ax^{1/3}$$

with $A = (3\pi)^{2/3}/2$. Making use of the relations (2.1), (2.2) we obtain

$$I(x) = \frac{\pi(1 - \sin\alpha)}{\cos\alpha - (\pi/2 - \alpha)\sin\alpha} - A\left(\frac{\pi\sin\alpha}{\cos\alpha - (\pi/2 - \alpha)\sin\alpha}\right)^{1/3}$$

with $0 \leqslant \alpha < \pi/2$ and we need to show that $I(x) > 0$ which is equivalent to

$$\Phi(\beta) = \frac{(1 - \cos\beta)^3}{\cos\beta[\sin\beta - \beta\cos\beta]^2} > \frac{9}{8}$$

where $\beta = \pi/2 - \alpha$. Since $\lim_{\beta \to 0} \Phi(\beta) = 9/8$ it is sufficient to show that $\Phi(\beta)$ increases and to this end we introduce the function

$$\begin{aligned}\Phi_1(\beta) &= \frac{\Phi'(\beta)}{\Phi(\beta)} \frac{(1 - \cos\beta)(\sin\beta - \beta\cos\beta)}{\sin\beta}\\ &= \operatorname{tg}\beta + 2\sin\beta - 3\beta.\end{aligned}$$

Now $\Phi_1(0) = \Phi_1'(0) = 0$ and $\Phi_1''(\beta) > 0$, then $\Phi_1(\beta) > 0$ and consequently $\Phi'(\beta) > 0$, i.e., the function $\Phi(\beta)$ increases and the inequality (3.2) follows.

To prove the asymptotic representation (3.1) we shall establish an asymptotic relation between x and $\beta = \pi/2 - \alpha$ for large values of x.

Recalling that $\alpha(x) \to \pi/2$ as $x \to \infty$, we get that $\beta = \beta(x)$ tends to zero. Thus from (2.1) we obtain

$$\frac{x}{\pi} = \frac{\cos\beta}{\sin\beta - \beta\cos\beta} = \frac{3}{\beta^3}\left[1 - \frac{2}{5}\beta^2 + O(\beta^4)\right], \qquad \beta \to 0,$$

from which

$$\beta = \left(\frac{3\pi}{x}\right)^{1/3} - \frac{2}{15}\frac{3\pi}{x} + O(x^{-5/3}), \qquad x \to \infty.$$

By using (2.2) and the last asymptotic representation of β we get

$$\begin{aligned} \iota(x) &= \frac{x}{\cos\beta} = \frac{x}{1 - \beta^2/2 + \beta^4/24 + O(\beta^6)} \\ &= x + \frac{(3\pi)^{2/3}}{2} x^{1/3} + \frac{3}{40}(3\pi)^{4/3} x^{-1/3} + O(x^{-1}), \qquad x \to \infty, \end{aligned}$$

and the proof of the theorem is complete.

Remark 3.1. Numerical computations indicate that the function $I(x) = \iota(x) - x - A(x^{1/3})$ decreases with $x > 0$, but we have not been able to prove this property.

4. Further Inequalities and Monotonic Results

From the concavity of the function $\iota(x)$ we obtain the simple inequality

$$\iota(x) \leqslant \iota(0) + \iota'(0)x, \qquad x > -1,$$

and using (2.2) and (2.11)

$$\iota(x) \leqslant \pi + (\pi/2)x, \qquad x > -1,$$

where the equality holds only in the case $x = 0$.

Thus from (2.4) we can write the general inequality

$$j_{\nu\kappa} < \kappa\pi + (\pi/2)\nu, \qquad \nu > -\kappa. \tag{4.1}$$

We observe that this inequality provides a better estimate in the case $-\frac{1}{2} < \nu < 0$ than McMahon's in (1.6).

On the other hand we know [4] that for $\nu > 0$ $c_{\nu k}$ is concave if $k = 2, 3, \ldots$

and $0 \leqslant \alpha < \pi$, or $k=1$, but $0 \leqslant \alpha \leqslant \pi/2$, which with our notations means that $j_{\nu\kappa}$ is concave with respect to ν if $\kappa \geqslant \frac{1}{2}$. Thus the inequality

$$j_{\nu\kappa} < j_{0\kappa} + \frac{d}{d\nu} j_{\nu\kappa}\big|_{\nu=0}\, \nu, \qquad \nu > 0, \tag{4.2}$$

follows.

In [1, p. 84] the upper bound $(d/d\nu) j_{\nu k} < \pi/2$ for $\nu > 0$ is given. Similarly we can obtain $(d/d\nu) j_{\nu\kappa} < \pi/2$ for $\nu > 0$ and all $j_{\nu\kappa}$, with the restriction $j_{\nu\kappa} > \nu$. In fact a result in [2] assures that if $0 \leqslant \nu < \infty$ and $j_{\nu\kappa} > \frac{1}{4}$, then $(d/d\nu) j_{\nu\kappa} > 1$ and $j_{\nu\kappa} > \nu + \frac{1}{4}$. Now, since $j_{0\kappa} > \frac{1}{4}$ we get $j_{\nu\kappa} > \nu$ and

$$\frac{d}{d\nu} j_{\nu\kappa}\big|_{\nu=0} < \frac{\pi}{2}$$

and from (1.6) $j_{0\kappa} < \kappa\pi$. Thus (4.2) is sharper than (4.1).

Moreover we observe that the concavity of $j_{\nu k}$, $k = 1, 2, \ldots$, has been proved in the general case $\nu > -k$ [1, p. 86]. Hence the inequality (4.2) is valid on the interval $(-k, 0)$ too.

Using the inequality (2.4) and the asymptotic representation of $\iota(x)$ given by (3.1) for $\nu/\kappa \to \infty$ we get

$$j_{\nu\kappa} < \nu + A\kappa^{2/3}\nu^{1/3} + B\kappa^{1/3}\nu^{-1/3} + O(\kappa^2/\nu)$$

with $A = (3\pi)^{2/3}/2$ and $B = (3/40)(3\pi)^{4/3}$.

A comparison between the last inequality and Tricomi's asymptotic formula (1.4) for fixed $\kappa = k$ and $\nu \to \infty$ gives

$$a_k k^{-2/3} < A, \qquad k = 1, 2, \ldots .$$

A direct computation of the left-hand side of the above expression, for $k = 1, 2, \ldots$, shows that the sequence $\{a_k k^{-2/3}\}_{k=1}^{\infty}$ increases and $a_k k^{-2/3} \to A$ as $k \to \infty$.

It would be of interest to show that the function $a_\kappa \kappa^{-2/3}$ increases with κ.

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